Characterization of Generalized Necessity Functions in Łukasiewicz Logic

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Abstract. We study a generalization of necessity functions to MV-algebras. In particular, we are going to study belief functions whose associated mass assignments have nested focal elements. Since this class of belief functions coincides with necessity functions on Boolean algebras, we will call them *generalized necessity functions*. Using geometrical and combinatorial techniques we provide several characterizations of these functions in terms of Choquet integral, Lebesgue integral, and min-plus polynomials.

Keywords: Necessity function, Belief function, MV-algebra

1 Introduction

There are at least two different, yet equivalent, ways to define necessities on Boolean algebras [4]. If the Boolean algebra is the set 2^X of all subsets of a given universe X, then the first approach consists in axiomatizing a necessity $N : 2^X \to [0,1]$ as a map satisfying $N(X) = 1$, $N(\emptyset) = 0$, and $N(A \cap B) = \min\{N(A), N(B)\}$. According to the second way, a necessity is viewed as a *belief function* [15] defined by a mass assignment $\boldsymbol{\mu} : 2^X \to [0,1]$ such that the class of its *focal elements* $\{ A \subseteq X \mid \boldsymbol{\mu}(A) > 0 \}$ is a chain with respect to the set inclusion. Since the former axiomatic approach can be traced back to Halpern's *belief measures* [6], we will henceforth distinguish

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between *necessity measures*, if the former is the case, and *necessity functions* in the latter case. These two ways of introducing necessities on Boolean algebras are equivalent. Specifically, a map $N : 2^X \to [0,1]$ is a necessity measure if and only if N is a necessity function.

Since MV-algebras [1] are among important many-valued generalizations of Boolean algebras, which provide a useful algebraic framework to deal with a certain and a relevant class of fuzzy sets, it is natural to ask what happens when we generalize necessity measures and necessity functions to these algebraic structures. Moreover, it is worth noticing that, as it was already remarked in [3], the generalizations of necessity measures and necessity functions to MV-algebras do not lead to one single concept as in the Boolean case. Hence it makes sense to study those notions separately.

In [5] the authors provide an axiomatic approach to necessity measures on MV-algebras and they show that they are representable by generalized Sugeno integrals. In this paper we characterize generalized necessity functions in the framework of the generalization of belief functions to MV-algebras proposed in [12]. In particular, we are going to use geometrical and combinatorial tools to provide several characterizations for these measures in terms of Choquet integral, Lebesgue integral, and min-plus polynomial.

The paper is organized as follows. In Section 2 we introduce the preliminaries about MV-algebras and states. We recall the theory of belief functions on Boolean algebras together with the equivalence between the two approaches to necessities in Section 3. Section 4 introduces generalized necessity functions with the main characterization (Proposition 3). Due to lack of space we are unable to include proofs; however, we provide examples to clarify main features of the discussed concepts.

2 Basic Notions

MV-algebras [1] play the same role for Łukasiewicz logic as Boolean algebras for the classical two-valued logic. An *MV-algebra* is an algebra $(M, \oplus, \neg, 0)$, where M is a non-empty set, the algebra $(M, \oplus, 0)$ is an abelian monoid, and these equations are satisfied for every $x, y \in M$: $\neg\neg x = x$, $x \oplus \neg 0 = \neg 0$, $\neg(\neg x \oplus y) = \neg(\neg y \oplus x)$.

In every MV-algebra M, we define the constant $1 = \neg 0$ and the following binary operations: for all $x, y \in M$, put $x \odot y = \neg(\neg x \oplus \neg y)$, $x \vee y = \neg(\neg x \oplus y) \oplus y$, $x \wedge y = \neg(\neg x \vee \neg y)$. For every $x, y \in M$, we write $x \leq y$ iff $\neg x \oplus y = 1$ holds in M. As a matter of fact, $\leq$ is a partial order on M, and M is said to be *linearly ordered* whenever $\leq$ is a linear order.

Example 1. Every Boolean algebra A is an MV-algebra in which the operations $\oplus$ and $\vee$ coincide (similarly, the operations $\odot$ and $\wedge$ coincide). Moreover, in every MV-algebra M, the set $B(M) = \{\, x \mid x \oplus x = x \,\}$ of its idempotent elements is the domain of the largest Boolean subalgebra of M (the so-called *Boolean skeleton* of M).

Example 2. Endow the real unit interval $[0,1]$ with the operations $x \oplus y = \min\{1, x+y\}$ and $\neg x = 1 - x$. Then $([0,1], \oplus, \neg, 0)$ becomes an MV-algebra called the *standard MV-algebra.* In this algebra, $x \odot y = \max\{0, x+y-1\}$, $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$. The two operations $\odot$, $\oplus$ are the so-called *Łukasiewicz t-norm* and the *Łukasiewicz t-conorm*, respectively.

Example 3. Let X be a nonempty set. The set $[0,1]^X$ of all functions $X \to [0,1]$ with the pointwise operations of the MV-algebra $[0,1]$ is an MV-algebra. In particular, if X is a finite set, say $X = \{1, \ldots, n\}$, then we can identify the MV-algebra $[0,1]^X$ with the n-cube $[0,1]^n$ and each $a \in [0,1]^X$ with the n-dimensional vector $a = (a_1, \ldots, a_n) \in [0,1]^n$. The set of vertices of $[0,1]^n$ coincides with the Boolean skeleton of $[0,1]^n$.

Throughout the paper, we will assume that X is always finite whenever we write $[0,1]^X$. The MV-algebra $[0,1]^X$ is the natural algebraic framework for studying belief functions in Łukasiewicz logic (cf. [10]). The extensions towards infinite X are possible and mathematically nontrivial (see [11, 12]). Herein we confine to the case of finite X for the sake of clarity.

Normalized and additive maps on MV-algebras (so-called states) were introduced in [7, 13]. States are many-valued analogues of probabilities on Boolean algebras. A *state* on an MV-algebra M is a function $s : M \to [0,1]$ satisfying the following properties:

(i) $s(0) = 0$, $s(1) = 1$,
(ii) $s(x \oplus y) = s(x) + s(y)$, whenever $x \odot y = 0$.

Observe that the restriction of every state s on M to its Boolean skeleton $B(M)$ is a finitely additive probability measure on $B(M)$. Much more is known: every MV-algebra M is (isomorphic to) an MV-algebra of continuous functions over some compact Hausdorff space X (see [1]) and each state on M is the Lebesgue integral with respect to a unique regular Borel probability measure on X (see [8] or [14]). In case of the MV-algebra $[0,1]^X$ with finite X, the previous fact can be formulated as follows. Observe that every probability measure on 2^X with $X = \{1, \ldots, n\}$ can be represented by a unique vector μ from the *standard n-simplex* $\Delta_n = \{\, \mu \in \mathbb{R}^n \mid \mu_i \geq 0,\ \sum_{i=1}^n \mu_i = 1 \,\}$.

Proposition 1 ([8,14]). *Let $X = \{1, \ldots, n\}$. If s is a state on $M = [0,1]^X$, then there exists a unique $\mu \in \Delta_n$ such that*

$$s(a) = \sum_{i=1}^{n} a_i \mu_i, \quad \text{for each } a \in M.$$

Moreover, the coordinates of μ are $\mu_i = s(\{i\})$, provided $\{i\}$ is identified with its characteristic function, for each $i \in X$.

3 Necessity Functions

See [15] for an in-depth treatment of Dempster-Shafer theory of belief functions. Let X be a finite set and $M = 2^X$. A *mass assignment* $\boldsymbol{\mu}$ is a function

$2^X \to [0,1]$ satisfying $\boldsymbol{\mu}(\emptyset) = 0$ and $\sum_{A\in 2^X} \boldsymbol{\mu}(A) = 1$. A *belief function* (with the mass assignment $\boldsymbol{\mu}$) is a function $\mathrm{Bel} : 2^X \to [0,1]$ given by $\mathrm{Bel}(A) = \sum_{B\subseteq A} \boldsymbol{\mu}(B)$, for each $A \in 2^X$. Each $A \in 2^X$ with $\boldsymbol{\mu}(A) > 0$ is said to be a *focal element*. A mass assignment $\boldsymbol{\mu}$ is called *nested* provided the set of its focal elements $\left\{ A \in 2^X \mid \boldsymbol{\mu}(A) > 0 \right\}$ is a chain in 2^X with respect to the set inclusion. By definition every belief function Bel is uniquely determined by the restriction of its mass assignment to the set of all focal elements. A *necessity function* on 2^X is a belief function whose mass assignment is nested. This can be rephrased as follows: a belief function is a necessity function iff its mass assignment determines a finitely additive probability on 2^{2^X} that is supported by a chain and vanishing at the singleton $\{\emptyset\}$.

If Bel is a belief function on 2^X, then the *credal set* of Bel is the following set $\mathcal{C}(\mathrm{Bel})$ of finitely additive probability measures P on 2^X:

$$\mathcal{C}(\mathrm{Bel}) = \left\{ P \mid P(A) \geq \mathrm{Bel}(A),\ A \in 2^X \right\}.$$

It is well-known that Bel arises as the lower envelope of $\mathcal{C}(\mathrm{Bel})$:

$$\mathrm{Bel}(A) = \bigwedge_{P\in\mathcal{C}(\mathrm{Bel})} P(A), \quad \text{for each } A \in 2^X.$$

Example 4. Let $A \in 2^X$ be nonempty and put $\mathrm{Bel}_A(B) = 1$, if $A \subseteq B$, and $\mathrm{Bel}_A(B) = 0$, otherwise. Then Bel_A is a necessity function whose mass assignment is

$$\boldsymbol{\mu}_A(B) = \begin{cases} 1, & A = B, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

The credal set $\mathcal{C}(\mathrm{Bel}_A)$ is just the set of all probabilities whose support is the set A. Specifically, this means that $\mathcal{C}(\mathrm{Bel}_A)$ is (affinely isomorphic to) the simplex $\Delta_{|A|}$, where $|A|$ is the cardinality of A. Observe that $A \subseteq B$ iff $\mathcal{C}(\mathrm{Bel}_A) \subseteq \mathcal{C}(\mathrm{Bel}_B)$ iff $\Delta_{|A|} \subseteq \Delta_{|B|}$.

In the next proposition we summarize some of the characterizations of necessity functions that appeared in the literature. Our goal is to compare these descriptions with the properties of extensions of necessity functions to MV-algebras in Section 4.

Proposition 2. *Let* Bel *be a belief function on* 2^X *with the mass assignment* $\boldsymbol{\mu}$. *Then the following are equivalent:*

(i) Bel *is a necessity function,*
(ii) $\mathrm{Bel}(A \cap B) = \mathrm{Bel}(A) \wedge \mathrm{Bel}(B)$, *for each* $A, B \in 2^X$,
(iii) the set $\left\{ \mathcal{C}(\mathrm{Bel}_A) \mid A \in 2^X,\ \boldsymbol{\mu}(A) > 0 \right\}$ *is a chain and*

$$\mathcal{C}(\mathrm{Bel}) = \sum_{\substack{A\in 2^X \\ \boldsymbol{\mu}(A)>0}} \boldsymbol{\mu}(A)\mathcal{C}(\mathrm{Bel}_A), \tag{2}$$

where the sum and the multiplication in (2) *are the Minkowski sum of sets and the pointwise multiplication of sets of vectors, respectively,*

(iv) *there exist* $n \in \{1, \dots, |X|\}$*, a vector* $\alpha = (\alpha_1, \dots, \alpha_n) \in \Delta_n$ *with all coordinates positive, and a chain of standard simplices* $\Delta_{i_1} \subset \dots \subset \Delta_{i_n}$*, where* $i_n \leq n$*, such that* $\mathcal{C}(\mathrm{Bel})$ *is (affinely isomorphic to) the Minkowski sum* $\sum_{j=1}^{n} \alpha_j \Delta_{i_j}$.

The equivalence of (i) with (ii) was proven in [15]. The properties (iii)-(iv) are a purely geometrical way to describe necessities by the composition of the associated credal sets. This approach has appeared first in [9], where the equivalence of (i) with (iii) was proven in a slightly more general setting. The property (iv) is just a direct reformulation of (iii). Geometrical treatment of belief functions appeared also in [2], where the properties of the set of all belief functions are discussed.

4 Generalized Necessity Functions

We will introduce the generalized necessity functions as particular cases of generalized belief functions in Łukasiewicz logic (cf. [10]). The starting point for this research was the generalization of Möbius transform established in a fairly general framework [11]. The interested reader is referred to those papers for further motivation and details.

If $X = \{1, \dots, n\}$, then by $\mathcal{P}$ we denote the set $2^X \setminus \{\emptyset\}$. Let $M_{\mathcal{P}}$ be the MV-algebra of all functions $\mathcal{P} \to [0, 1]$. We will consider the following embedding ρ of the MV-algebra $M = [0, 1]^n$ into $M_{\mathcal{P}}$:

$$\rho : M \times \mathcal{P} \to [0, 1], \;\; \rho_a(A) = \bigwedge_{i \in A} a_i, \quad \text{for each } a \in M,\, A \in \mathcal{P}.$$

If $a \in M$ is fixed and $\rho_a(\emptyset) := 0$, then observe that function $\rho_a : 2^X \to [0, 1]$ is a necessity measure on 2^X.

Definition 1. Let M be the MV-algebra $[0, 1]^X$. A *state assignment* is a state $\mathbf{s}$ on $M_{\mathcal{P}}$. If $\mathbf{s}$ is a state assignment, then a *(generalized) belief function* Bel^* on M is given by $\mathrm{Bel}^*(a) = \mathbf{s}(\rho_a)$, $a \in M$. We say that a belief function Nec^* on M is a *(generalized) necessity function* if the finitely additive probability on $2^{2^{\mathcal{P}}}$ corresponding to its state assignment (via Proposition 1) is supported by a chain.

Example 5. Let $A \in \mathcal{P}$ and put $\mathrm{Bel}^*_A(a) = \rho_a(A)$. Clearly, function Bel^*_A is a necessity function. Its state assignment $\mathbf{s}_A$ is given by $\mathbf{s}_A(f) = f(A)$, for each $f \in M_{\mathcal{P}}$.

Remark 1. Following the analogy with Proposition 2(ii), necessity measures on an MV-algebra M have been recently introduced in [5] as mappings $N : M \to [0, 1]$ such that $N(1) = 1$, $N(0) = 0$, and for every $a, b \in M$, $N(a \wedge b) = N(a) \wedge N(b)$. It was observed already in [3] by Dubois and Prade that, in sharp contrast with the classical case (cf. Proposition 2), necessity functions are not necessity measures. Indeed, generalized necessity functions do not satisfy the property $N(a \wedge b) = N(a) \wedge N(b)$, in general: this follows directly from Definition 1.

Let Bel^* be a belief function on $M = [0,1]^X$ and let $\mathbf{s}$ be its associated state assignment. Clearly, for each $A \in \mathcal{P}$, the mass assignment $\boldsymbol{\mu}_A$ from (1) is an element of $M_{\mathcal{P}}$. As a direct consequence of the definition of state, the function $\boldsymbol{\mu}_{\mathbf{s}} : 2^X \to [0,1]$ defined by $\boldsymbol{\mu}_{\mathbf{s}}(A) = \mathbf{s}(\boldsymbol{\mu}_A)$ for every $A \in \mathcal{P}$, and zero otherwise, is a mass assignment. Hence $\mathrm{Bel}^*(a) = \sum_{A \in M_{\mathcal{P}}} \rho_a(A) \boldsymbol{\mu}_{\mathbf{s}}(A)$, for each $a \in M$, which follows from Proposition 1.

If Bel^* is a belief function on M, then the *credal set* of Bel^* is the following set $\mathcal{C}(\mathrm{Bel}^*)$ of states s on M:

$$\mathcal{C}(\mathrm{Bel}^*) = \{\, s \mid s(a) \geq \mathrm{Bel}^*(a),\, a \in M \,\}\,.$$

It can be shown that Bel^* is the lower envelope of $\mathcal{C}(\mathrm{Bel}^*)$:

$$\mathrm{Bel}^*(a) = \bigwedge_{s \in \mathcal{C}(\mathrm{Bel}^*)} s(a), \quad \text{for each } a \in M. \tag{3}$$

In the following proposition we give several equivalent formulations describing generalized necessity functions within the class of generalized belief functions. In particular, some of the properties directly correspond to the respective properties of necessity functions—see Proposition 2.

Proposition 3 (Characterization of generalized necessity functions). *Let $X = \{1, \ldots, n\}$ and Bel^* be a belief function on the MV-algebra $M = [0,1]^n$ with the state assignment $\mathbf{s}$ and the mass assignment $\boldsymbol{\mu}_{\mathbf{s}}$. Then the following are equivalent:*

(i) Bel^* *is a necessity function,*

(ii) there exists a necessity measure Nec *on 2^X such that*

$$\mathrm{Bel}^*(a) = \oint a \,\mathrm{d}\,\mathrm{Nec}, \quad a \in M,$$

where the discrete integral above is the Choquet integral,

(iii) the mass assignment $\boldsymbol{\mu}_{\mathbf{s}}$ is nested on a chain $\mathcal{A} \subseteq \mathcal{P}$ such that

$$\mathrm{Bel}^*(a) = \sum_{A \in \mathcal{A}} \boldsymbol{\mu}_{\mathbf{s}}(A) \rho_a(A), \quad a \in M,$$

(iv) the mass assignment $\boldsymbol{\mu}_{\mathbf{s}}$ is nested on a chain $A_1 \subset \cdots \subset A_k$ such that

$$\mathrm{Bel}^*(a) = \bigwedge_{(i_1, \ldots, i_k) \in I} \sum_{j=1}^{k} \boldsymbol{\mu}_{\mathbf{s}}(A_j) a_{i_j}, \quad a \in M,$$

where $I = A_1 \times \cdots \times A_k$, *(v) there exists a maximal chain $\mathcal{A} = A_1 \subset A_2 \subset \cdots \subset A_n = X$ in 2^X and a mass assignment $\boldsymbol{\mu}$ nested on $\mathcal{A}$ such that*

$$\mathrm{Bel}^*(a) = \bigwedge_{s=1}^{n!} \left(\sum_{i=1}^{n} \boldsymbol{\mu}(A_i) \cdot a_{f^{-1}(s)(i)} \right), \quad a \in M,$$

where $f : A_1 \times \ldots \times A_n \to \{1, 2, \ldots, n!\}$ is a bijection,

(vi) the set $\left\{ \mathcal{C}(\mathrm{Bel}^*_A) \;\middle|\; A \in 2^X, \boldsymbol{\mu}_{\mathbf{s}}(A) > 0 \right\}$ is a chain and

$$\mathcal{C}(\mathrm{Bel}^*) = \sum_{\substack{A \in 2^X \\ \boldsymbol{\mu}_{\mathbf{s}}(A) > 0}} \boldsymbol{\mu}_{\mathbf{s}}(A) \mathcal{C}(\mathrm{Bel}^*_A),$$

(vii) there exist $n \in \{1, \ldots, |X|\}$, a vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in \Delta_n$ with each $\alpha_i \geq 0$, and a chain of standard simplices $\Delta_{i_1} \subset \cdots \subset \Delta_{i_n}$, where $i_n \leq n$, such that $\mathcal{C}(\mathrm{Bel}^*)$ is (affinely isomorphic to) the Minkowski sum $\sum_{j=1}^{n} \alpha_j \Delta_{i_j}$.

Proposition 3, whose proof is omitted due to a lack of space, provides a number of interpretations of necessity functions. In particular, (ii) means that each generalized necessity function is recovered as the Choquet integral extension of a necessity measure. The properties (vi)-(vii) say that the credal set of a generalized necessity function is built from "nested" simplices in a very special way—observe that this is identical with the property of necessity functions on Boolean algebras (Proposition 2(iii)-(iv)). The min-sum formula in (iv) is then a consequence of this geometrization together with (3): when minimizing a linear function given by $a \in [0,1]^n$ over $\mathcal{C}(\mathrm{Bel}^*)$, it suffices to seek the minimum among the elements of any finite set containing the vertices of the convex polytope $\mathcal{C}(\mathrm{Bel}^*)$. Notice that although the equivalence between (iv) and (v) is clear, because in fact (v) is a particular case of (iv), (v) can be easily proved to be equivalent to (iii) by using a combinatorial argument. The results are illustrated with a simple example.

Example 6. Let $X = \{1, 2, 3\}$ and $M = [0,1]^X$. Suppose that Nec is the necessity measure on 2^X whose mass assignment $\boldsymbol{\mu}$ is defined as $\boldsymbol{\mu}(\{1\}) = \frac{1}{8}$, $\boldsymbol{\mu}(\{1,2\}) = \frac{4}{8}$, $\boldsymbol{\mu}(X) = \frac{3}{8}$. The necessity function Nec^* associated with Nec via Proposition 3(ii) is then

$$\mathrm{Nec}^*(a) = \tfrac{1}{8} a_1 + \tfrac{4}{8}(a_1 \wedge a_2) + \tfrac{3}{8}(a_1 \wedge a_2 \wedge a_3),$$

for each $a \in [0,1]^3$. Due to Proposition 3(vi), the credal set $\mathcal{C}(\mathrm{Nec}^*)$ can be identified with the Minkowski sum $\frac{1}{8}\Delta_1 + \frac{4}{8}\Delta_2 + \frac{3}{8}\Delta_3$. This is a convex polytope embedded in Δ_3 with the four vertices $(1,0,0)$, $(\frac{1}{8}, \frac{7}{8}, 0)$, $(\frac{5}{8}, 0, \frac{3}{8})$, and $(\frac{1}{8}, \frac{4}{8}, \frac{3}{8})$. This means together with Proposition 3(v) that we get the min-sum formula

$$\mathrm{Nec}^*(a) = a_1 \wedge \left(\tfrac{1}{8}a_1 + \tfrac{7}{8}a_2\right) \wedge \left(\tfrac{5}{8}a_1 + \tfrac{3}{8}a_3\right) \wedge \left(\tfrac{1}{8}a_1 + \tfrac{4}{8}a_2 + \tfrac{3}{8}a_3\right).$$

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